The n-Dimensional Cube and the Tower of Hanoi
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Using the following inequalities the stated inequality is proved:

\[ x_2 - x_1 \geq l - \frac{\pi}{\sqrt{k}}, \text{ when } k \geq \frac{\pi^2}{l^2} \]

\[ \frac{u'}{u} \bigg|_{x_2}^{x_1} \geq 2\sqrt{k} \cot \frac{\sqrt{k} l}{2}, \text{ when } k < \frac{\pi^2}{l^2} \]

(the latter of these can be proved by elementary differentiation).

**Remark.** In the case \( k < \pi^2/l^2 \) one can see the inequality (1) perhaps still more simply in the following manner: Then a Green's function \( G(x, z) \) of the problem \( y'' + ky = 0, \ y(0) = y(l) = 0 \) exists and we have

\[ y(x) = \int_0^l G(x, z)(y''(z) + ky(z))dz. \]

\( G(x, z) \) does not change sign within \( 0 < x, z < l \). One obtains by differentiation

\[ M = \max_{0 \leq x, z \leq l} |G(x, z)| = \frac{1}{2\sqrt{k} \cot \frac{\sqrt{k} l}{2}}. \]

If one defines \( m = \max_{0 \leq x \leq l} y(x) = y(x_1) \) it follows that

\[ T(y, k) \geq \frac{1}{mM} \left| \int_0^l G(x_1, z)(y''(z) + ky(z))dz \right| = \frac{1}{M} = 2\sqrt{k} \cot \frac{\sqrt{k} l}{2}. \]

**The n-Dimensional Cube and the Tower of Hanoi**

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The purpose of this note is to point out a connection between a Hamiltonian circuit, abbreviated H-circuit, of the n-cube and the moves of the well known tower of Hanoi puzzle.*

With respect to a rectangular coordinate system in Euclidean n-space, let the vertices of an n-cube be defined by the \( 2^n \) ordered n-tuples, \((\varepsilon_1, \ldots, \varepsilon_n)\), \( \varepsilon_i = 0, 1 \) for all \( i \). By the 1-skeleton of the n-cube is meant the linear graph whose vertices and lines are the vertices and edges of the n-cube. An H-circuit of a linear graph is a closed path containing each vertex exactly once. An H-circuit of the n-cube is defined to be an H-circuit of its 1-skeleton. A sequence of \( 2^n \) vertices, \( v_1, \ldots, v_{2^n} \), of the n-cube is said to **describe** an H-circuit of the n-cube if the closed path \( v_1v_2 \cdots v_{2^n}v_1 \) is an H-circuit of the n-cube.

The following two remarks are easily verified. (1) Two vertices,

* See, for example, W. W. Rouse Ball, Mathematical Recreations and Essays, Rev. ed., New York, 1947, pp. 262–266 (Hamiltonian circuit), 303–305 (tower of Hanoi); H. S. M. Coxeter, Regular Polytopes, New York, 1949, pp. 8 (Hamiltonian circuit), 123 (n-cube).
v = (ε₁, ⋅⋅⋅, εₙ) and v' = (η₁, ⋅⋅⋅, ηₙ), of the n-cube are connected by an edge, vv', if and only if there is some k, 1 ≤ k ≤ n, such that εᵢ ≠ ηₖ and εᵢ = ηᵢ for all i ≠ k. (2) Any sequence of 2ⁿ different n-tuples of 0's and 1's having the property that any two consecutive n-tuples differ in exactly one entry (where the first and last n-tuples are to be considered as consecutive) describes an H-circuit of the n-cube.

We now describe briefly the tower of Hanoi. This puzzle consists of three pegs, on one of which are piled n flat pierced disks, each smaller than the one below it. The problem is to use the fewest possible moves to transfer these disks to one of the other two pegs by removing one disk at a time from the top of any pile and placing it on any peg which does not already hold a smaller disk. It is easily seen that there is a (unique) solution to the problem requiring exactly 2ⁿ−1 moves. That is, counting the initial position and the final position, there are 2ⁿ different positions of the disks. The sequence of positions, in the order in which they occur, is called an Hₙ-sequence. We propose to identify these positions with the vertices of the n-cube. Let the disks be numbered according to increasing size with the numbers 1, ⋅⋅⋅, n. Let the jth position of the disks in the solution of the puzzle be represented by the n-tuple of 0's and 1's, (εᵢⱼ, ⋅⋅⋅, εᵢⱼ, ⋅⋅⋅, εᵢⱼ), where εᵢⱼ is the number of moves, modulo 2, which have been completed by the jth disk.

**Theorem.** The sequence of n-tuples in an Hₙ-sequence describes an H-circuit of the n-cube.

**Proof.** We have only to show that an Hₙ-sequence has the properties given in (2). Except possibly for the first and last n-tuples of the sequence it is clear that any two consecutive n-tuples differ in exactly one entry, for any position is obtained from its predecessor by a move of exactly one disk. The first and last n-tuples are also so related, for the first consists entirely of 0's and the last consists entirely of 0's except for a final 1. It remains only to prove that all the terms of the sequence are different. For n = 1 this is clearly true. The general case is proved by an inductive argument, using the fact that for n + 1 disks one first solves the puzzle for n disks, then moves the largest disk (number n + 1) and replaces the n disks on top of it by re-solving the puzzle for n disks. Consequently, the Hₙ₊₁-sequence contains (in a rearrangement) the 2ⁿ different terms of the Hₙ-sequence augmented by a 0 in the (n + 1)st place and the same 2ⁿ different terms augmented by a 1 in the (n + 1)st place. Thus the terms of the Hₙ₊₁-sequence are all different. This completes the proof of the theorem.

We conclude with the following remark. Suppose the jth step in solving the Hanoi puzzle consists in moving the kth disk, clockwise or counterclockwise according as k is even or odd. Then the successive values of k are 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, ⋅⋅⋅. In fact, k is one more than the number of factors 2 in j. That is, k is given by the formula j = 2ᵏ−₁l, where l is odd. The sequence of k's facilitates the description of the H-circuit, for its jth edge is parallel to the kth coordinate axis.