9. M. Marden, On the zeros of rational functions having prescribed poles, etc. Trans. Amer. Math. Soc., 66 (1949) 407–418.

10. M. Biernacki, Sur le déplacement des zéros des fonctions entières par leur dérivation, C. R. Acad. Sci. Paris, 175 (1922) 18–20. Also, M. Alander, Sur les fonctions entières non-réelles, Ark. Mat., Astron. Fysik, 18 (1924) 1–9.

11. E. Borel, Leçons sur les fonctions entières, Gauthier-Villars, Paris, 1900.

12. M. Marden, On the derivative of an entire function, Proc. Amer. Math. Soc., 74 (1968).

ON PAVING THE PLANE

R. B. KERSHNER, Johns Hopkins University

One of the oldest of geometrical problems is the determination of those plane figures which share with the square and regular hexagon the ability to cover the plane, by congruent repetitions, without gaps or overlaps. This problem, called variously the problem of tesselation, plane tiling or plane paving, was brought anew into mathematical prominence by Hilbert [1] in 1900 when he posed it as one of his "Mathematische Probleme."

Particular interest centers on those plane figures which are convex polygons and even for this special case the problem poses ample difficulties. It is readily seen that all triangles and quadrilaterals do pave the plane. It can further be shown, with the help of Euler's equation for the vertices, edges and faces of a polygonal network, that no convex polygon with more than six sides can pave the plane. This has been demonstrated by a number of authors. So the problem reduces to the determination of those convex hexagons and pentagons which can pave the plane.

The author has shown that there are exactly three types of hexagons and eight types of pentagons (of which three are special cases of the three types of hexagons) which can pave the plane.

In order to state the results, let the angles of a hexagon be denoted, consecutively, as A, B, C, D, E, F, and the sides as a, b, c, d, e, f, in such a way that a and b are the sides of A, b and c are the sides of B, etc. Similarly, for a pentagon, let the angles be, consecutively, A, B, C, D, E, and the sides a, b, c, d, e, with a and b the sides of A, etc. Then we can state

THEOREM 1. *A convex hexagon can pave the plane if and only if it is of one of the following three types:*

Hexagon of Type 1: $A+B+C=2\pi$, $a=d$;

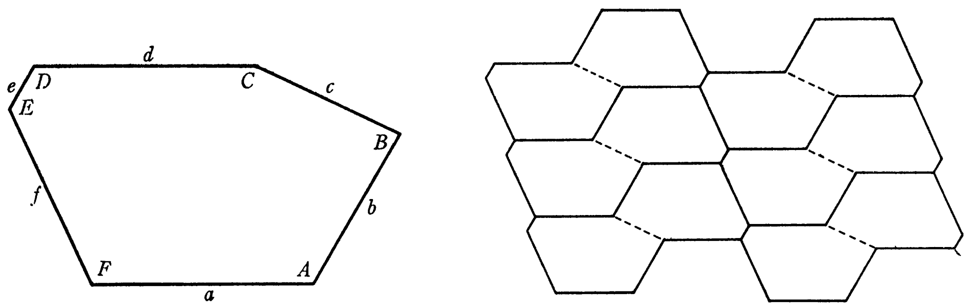
Hexagon of Type 2: $A+B+D=2\pi$, $a=d$, $c=e$;

Hexagon of Type 3: $A=C=E=\frac{2}{3}\pi$, $a=b$, $c=d$, $e=f$.

THEOREM 2. *A convex pentagon can pave the plane if and only if it is one of the following eight types:*

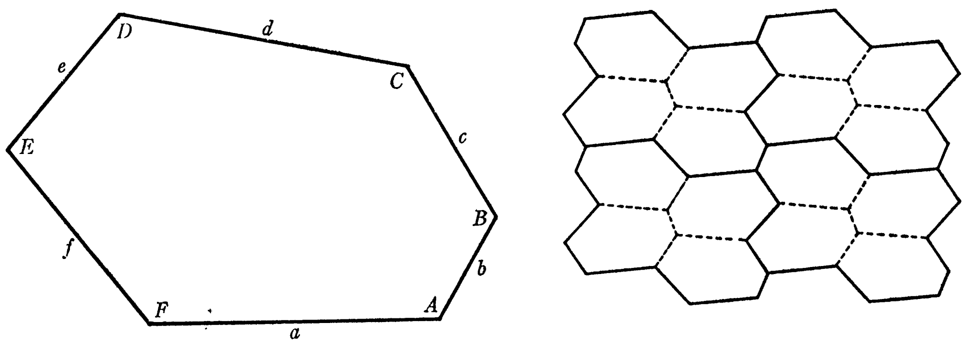
Pentagon of Type 1: $A+B+C=2\pi$;

Pentagon of Type 2: $A+B+D=2\pi$, $a=d$;



(a) Individual hexagon (b) Section of pavement

FIG. 1: Hexagon of Type 1; $A+B+C=2\pi$, $a=d$.



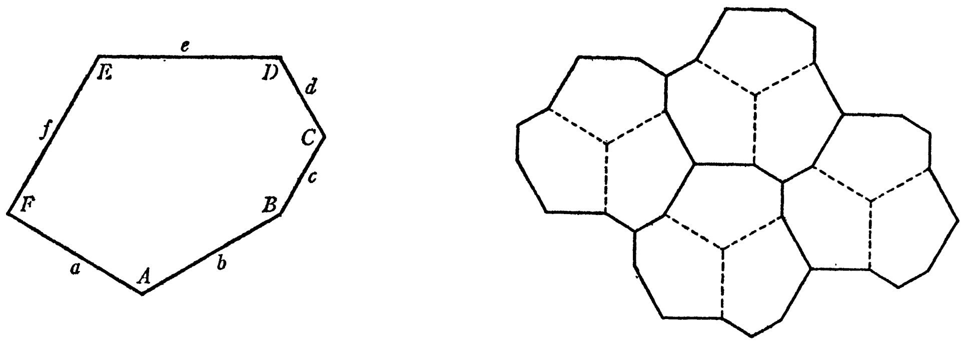
(a) Individual hexagon (b) Section of pavement

FIG. 2: Hexagon of Type 2; $A+B+D=2\pi$, $a=d$, $c=e$.

Pentagon of Type 3: $A=C=D=\frac{2}{3}\pi$, $a=b$, $d=c+e$;
Pentagon of Type 4: $A=C=\frac{1}{2}\pi$, $a=b$, $c=d$;
Pentagon of Type 5: $A=\frac{1}{3}\pi$, $C=\frac{2}{3}\pi$, $a=b$, $c=d$;
Pentagon of Type 6: $A+B+D=2\pi$, $A=2C$, $a=b=e$, $c=d$;
Pentagon of Type 7: $2B+C=2D+A=2\pi$, $a=b=c=d$;
Pentagon of Type 8: $2A+B=2D+C=2\pi$, $a=b=c=d$.

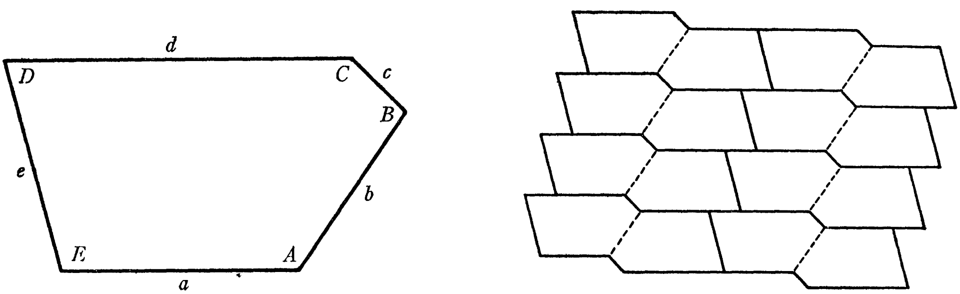
The first three types of pentagons can be considered as special cases of the three types of hexagons and indeed can be converted to hexagons of the desired types by appropriately inserting a vertex along one of the sides. However, the remaining five types of pentagons do not arise as special cases of hexagons which can pave the plane.

The proof that the list in Theorems 1 and 2 is complete is extremely laborious and will be given elsewhere. The fact that these types do pave, however, is quite straightforward and, indeed, is adequately indicated by the accompanying illustrative figures.



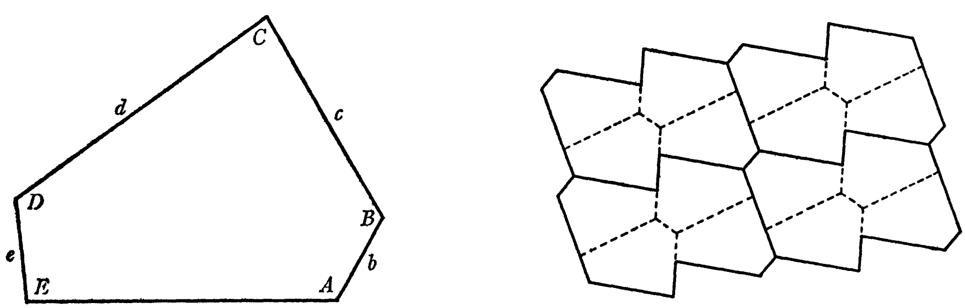
(a) Individual hexagon (b) Section of pavement

FIG. 3: Hexagon of Type 3; $A=C=E=\frac{2}{3}\pi$, $a=b$, $c=d$, $e=f$.



(a) Individual pentagon (b) Section of pavement

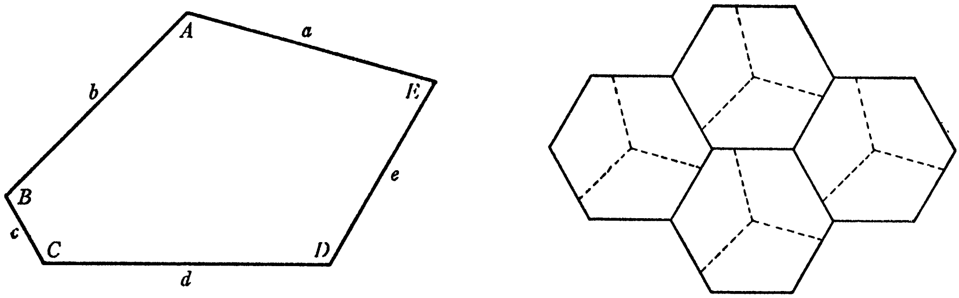
FIG. 4: Pentagon of Type 1; $A+B+C=2\pi$.



(a) Individual pentagon (b) Section of pavement

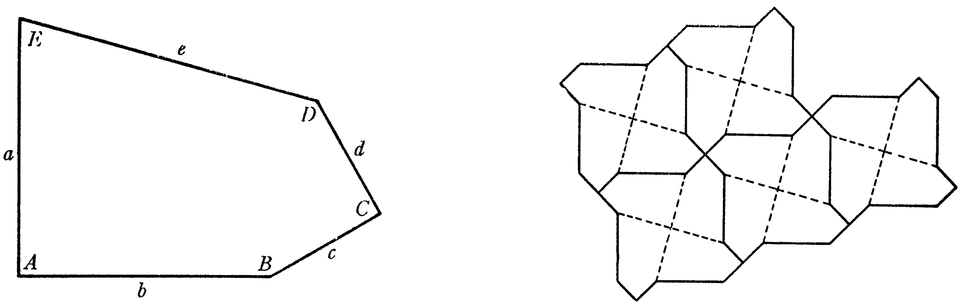
FIG. 5: Pentagon of Type 2; $A+B+D=2\pi$, $a=d$.

It should be noted that the problem of tesselation, not only for convex polygons but for quite general bounded figures, has been treated in detail in a recent book [2]. Indeed, it is stated that the treatment given is complete and that all figures which pave are derived from a general classification scheme developed



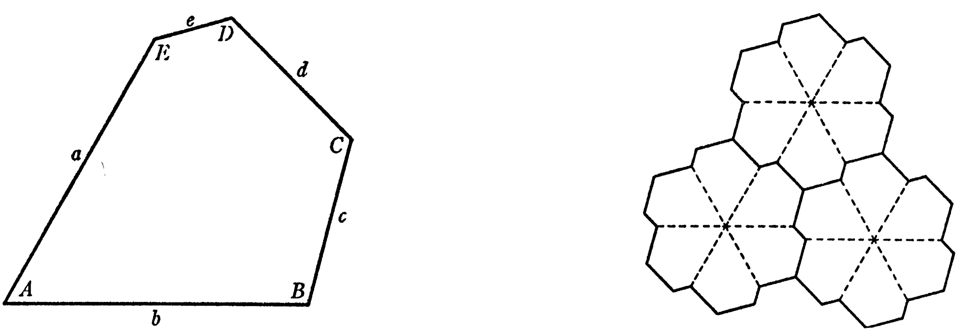
(a) Individual pentagon (b) Section of pavement

FIG. 6: Pentagon of Type 3; $A=C=D=\frac{2}{3}\pi$, $a=b$, $d=c+e$.



(a) Individual pentagon (b) Section of pavement

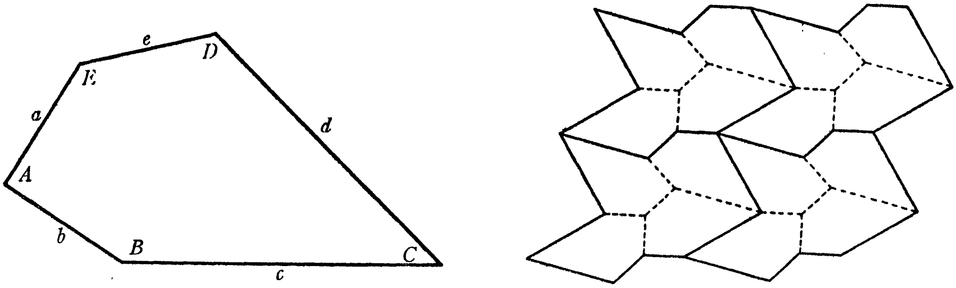
FIG. 7: Pentagon of Type 4; $A=C=\frac{1}{2}\pi$, $a=b$, $c=d$.



(a) Individual pentagon (b) Section of pavement

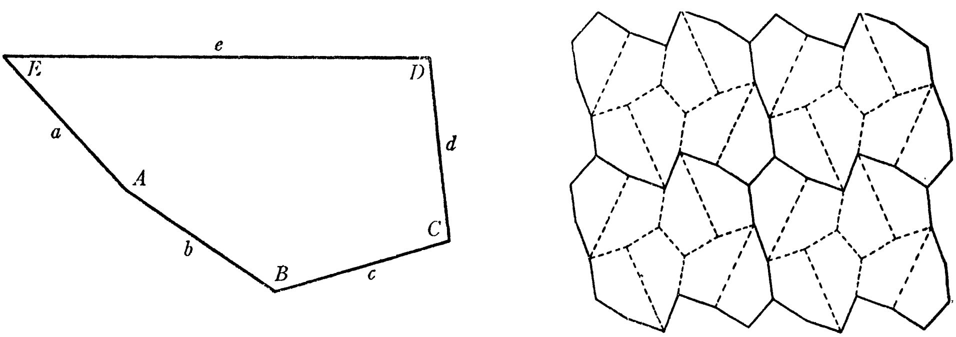
FIG. 8: Pentagon of Type 5; $A=\frac{1}{3}\pi$, $C=\frac{2}{3}\pi$, $a=b$, $c=d$.

by Heinrich Heesch in 1932. Unfortunately this is not the case. When applied to the particular case of convex polygons, the general classification scheme of Heesch yields the three hexagon types of Theorem 1 and the first five pentagon types of Theorem 2 but does not yield the pentagons of Types 6, 7, 8. The three



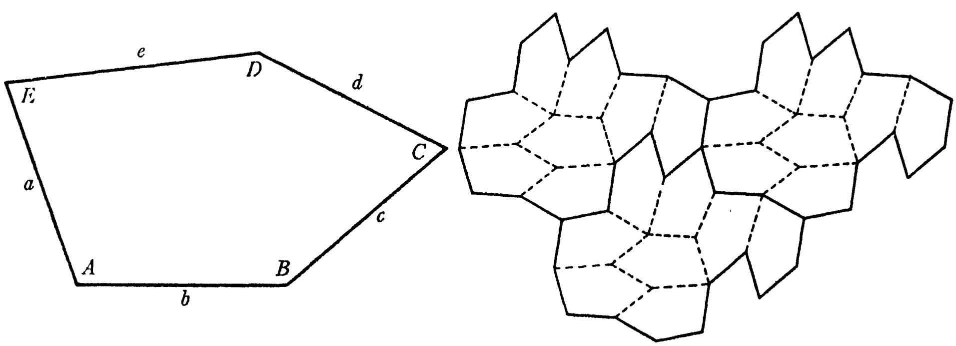
(a) Individual pentagon (b) Section of pavement

FIG. 9: Pentagon of Type 6; $A+B+D=2\pi$, $A=2C$, $a=b=e$, $c=d$.



(a) Individual pentagon (b) Section of pavement

FIG. 10: Pentagon of Type 7; $2B+C=2\pi$, $2D+A=2\pi$, $a=b=c=d$.



(a) Individual pentagon (b) Section of pavement

FIG. 11: Pentagon of Type 8; $2A+B=2\pi$, $2D+C=2\pi$, $a=b=c=d$.

hexagon pavings and the first five pentagon types have also been discovered independently by other authors [3, 4] as well. However, the paving by pentagons of Types 6, 7, 8 appears to be previously unknown.

References

1. D. Hilbert, Pariser Vortrag, "Mathematische Probleme," Problem 18. Nachr. Gesellsch. Wiss. Göttingen, Mathem.-physik. Klasse, 1900.
2. H. Heesch und O. Kienzle, Flächenschluss, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
3. K. Reinhardt, Über die Zerlegung der Ebene in Polygone, Dissertation der Naturwiss. Fakultät, Universität Frankfurt/Main, Borna 1918.
4. B. Bollobas, Filling the plane with congruent convex hexagons without overlapping, Annals Universitatis Scientiarum Budepestinensis, 1963.
5. ———, A sík lefedése egybevágó konvex sokszögekkel, I and II, Különlenyomat Középiskolai Matematikai Lapok, 1961.

SEPARATION OF VARIABLES

OTTO PLAAT, University of San Francisco

1. Introduction. Leibniz's method of separation of variables has had a history of singular neglect. Judging by the literature of the past fifty years, the method has at best been underestimated in respect of its range and power, and at worst has been seriously misunderstood. Considering that tradition has surrounded it with an aura of hesitation and ambiguity, it is not surprising that there is no consensus as to what precisely the method is: to which equations it applies, and what it does. (In lieu of a bibliography, we refer the reader to his three favorite texts.)

Separation of variables involves essentially two indefinite integrations and the inversion of a function. Whether these can be effected in elementary terms—that is, whether the solutions produced by the algorithm are elementary functions—is a question of differential algebra, and concerns us only indirectly. The results in which we are interested refer to the analytic aspect of the algorithm, and particularly to the fact that the method produces solutions in a certain form. The principal purpose of this note is to call attention to Theorem 3, which tells us among other things that whenever the algorithm is effective, i.e., produces elementary functions, the uniqueness or nonuniqueness character of the given differential equation can be deduced from an inspection of the solutions computed. We also append some remarks suggesting uses of the method for obtaining qualitative results in an introductory course on differential equations.

In order to emphasize the elementary and self-contained character of the subject, we shall use no results from the theory of differential equations whatever. It will be necessary to begin by indicating the precise sense in which we employ certain standard notions. The term "S solution" is defined in Section 3.

2. Definitions and Hypotheses. An *interval* will be a connected set in R containing more than one point. We consider the equation